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Free convection flow of a conducting couple stress fluid in a porous medium

A state space approach

G. Shantha
Department of Mathematics and Humanities, MGIT Hyderabad, Hyderabad, India, and
Bandari Shanker
Department of Mathematics, Nizam College, Hyderabad, India

Abstract

Purpose – This paper aims to examine the unsteady flow of an incompressible, viscous, conducting couple stress fluid flow through a porous medium over an infinite plate that is started into motion in its own plane by an impulse. The presence of a uniform magnetic field in a direction perpendicular to the plate is assumed.

Design/methodology/approach – Casting the governing equations into matrix form, the authors use state space approach and Laplace transform technique to obtain the field variables. The inversion of Laplace transform is carried through the adoption of a numerical technique.

Findings – Two specific problems related to a heated plate and a plate under uniform heating are examined and the variation of temperature and velocity with respect to the couple stress parameter numerically is studied. Numerical results concerning velocity and temperature distribution are presented graphically.

Originality/value – The authors have attempted the problem in fluid dynamics through state space approach which is exploited with tremendous success in modern control theory but not enough in fluid dynamics.

Keywords Fluid dynamics, Flow, Laplace transforms, Numerical analysis

Paper type Research paper

1. Introduction

It is well known that in many of the real fluid flows the shear behavior cannot be characterized by Newtonian theory. Hence during the last century several attempts were made to propose new constitutive equations for diverse fluids taking into account some aspects of certain real fluids. The stress-strain relationships in the newly stated theories given by the constitutive equations are no longer linear/Newtonian and hence these new ones are referred to as non-Newtonian fluid theories. One such theory is that of the couple stress fluids which was initiated by V.K. Stokes (1966) in 1966. Consider a body B enclosing a volume V without considering the microstructures of the infinitesimal fluid volume element. The set of all forces acting on an infinitesimal volume element δv are, in general, equivalent to a single resultant force together with a resultant couple. Let us assume that the moment of the couple is not zero. With this assumption V.K. Stokes has proposed the theory of couple stress fluids allowing for the sustenance of couple stresses in addition to the usual stresses. The fluids can also sustain the existence of body forces as usual and in addition body couples as well. The stress tensor is no longer symmetric in this theory. This is one among the several non-Newtonian fluid theories developed in the 20th century. This fluid theory

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is supposed to be a model for describing the behavior of lubricants in industrial problems or in physiology. Several boundary and initial value problems have been solved with reference to this fluid theory earlier and also in recent years (Stokes, 1984; Lakshman Rao and Iyengar, 1985; EL-Dabe and EL-Mohhsndis, 1995; Naduvinamanin *et al.*, 2003; Lin and Hung, 2005, 2007).

The flow of an electrically conducting couple stress fluid heated from below in the presence of a magnetic field was considered by Sharma and Thakur (2000). These authors made some interesting observations regarding the effect of couple stress field and magnetic field in the case of stationary convection. In this paper, we consider the magneto hydro dynamic free convection flow of a couple stress fluid in a porous medium past an infinite plate that is started into motion in its own plane by an impulse. We study this problem by adopting the method of state space formulation. The state space approach which has been hitherto applied to problems in modern control systems and allied fields has its basis in the matrix exponential method and is applied to any physical process where the behavior with respect to time is of interest. This approach is more general than the classical Laplace and Fourier Transform Techniques. In view of this, it is applicable to all systems that can be analyzed by integral transforms in time. An excellent exposition of the state space analysis of control systems is available in the classic book (Ogata, 1967) due to Ogata. The success of state space approach is due to the fact that linear systems with time variant parameters can be analyzed essentially in the same manner as in time invariant linear systems. The present study is motivated by the study of a similar problem attempted by Helmy *et al.* for the case of micropolar fluid (Helmy *et al.*, 2002). Very recently Devakar and Iyengar solved the Stokes' first problem for a micropolar fluid through state space approach (Devakar and Iyengar, 2009).

2. Statement of the problem and mathematical formulation

Consider the unsteady flow of an incompressible viscous conducting couple stress fluid through a porous medium of permeability K past an infinite plate that is started into motion in its own plane by an impulse. We consider the influence of a transversely applied magnetic field on the flow generated. Take a point O in space, through which the infinite plate is drawn in the fluid medium, as the origin. Let the normal to the plate through O be taken as y-axis and the direction of motion of the plate through O be taken as x-axis. It is possible that the motion of the conducting fluid may induce an electric current and this may distort the applied magnetic field. We assume that there arises no distortion. In many of the aerodynamic applications and for electrically conducting liquid metals this is true (Helmy *et al.*, 2002). We further assume that:

- The fluid viscosity μ, couple stress viscosity coefficient η and density ρ are all constants.
- The effects of dissipated energy are neglected.
- As a result of the application of generalized Fourier Law the effect of relaxation time τ_0 is considered.
- $B_0 = \mu_e H_0$ is the nonvanishing component of the magnetic induction and the Pondermotive force has one nonvanishing component which is along the x-direction given by

$$F_x = -\frac{\sigma B_0^2 u}{\rho}$$

In view of the nature of the problem, we have the velocity vector $\bar{q}$ and temperature T in the form:

$$\bar{q} = (u(y,t),\ v(y,t),\ 0); \quad T = T(y,t) \tag{2.1}$$

with the above assumptions, the unsteady magneto hydro dynamic free convection flow equations of a couple stress fluid are given by (see chapter 3 in Stokes, 1984):

$$\frac{\partial v}{\partial y} = 0 \text{ (Continuity equation)} \tag{2.2}$$

$$\frac{1}{\alpha}\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - e\frac{\partial^4 u}{\partial y^4} - \frac{\upsilon u}{K} - \frac{\sigma B_0^2 u}{\rho} \text{ (Linear momentum equation)} \tag{2.3}$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} + \tau_0\frac{\partial}{\partial t}\left[\frac{1}{\alpha}\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y}\right] = \frac{1}{\rho c_p}\left[\lambda\frac{\partial^2 T}{\partial y^2} + Q + \tau_0\frac{\partial Q}{\partial t}\right] \text{ (Energy equation)} \tag{2.4}$$

where

$$\upsilon = \frac{\mu}{\rho}; \quad e = \frac{\eta_1}{\rho}$$

$\alpha, \sigma, B_0, \lambda,$ and τ_0 are respectively porosity constant, electrical conductivity, magnetic induction, the thermal conductivity, and the thermal relaxation time.

In the above equations, u and v are the velocity components in the x and y directions respectively. T is the fluid temperature. μ and η are the viscosity and couple stress viscosity coefficients respectively as stated earlier. K as already mentioned earlier is permeability constant and Q is intensity of heat source.

Stokes (1984) proposes mainly two types of boundary conditions:

(1) the vorticity of the fluid on the boundary is equal to the rotational velocity of the boundary, and

(2) the couple stresses vanish on the boundary.

We propose to solve the present problem with the boundary condition (1). In view of this, the relevant boundary conditions are:

$$\begin{aligned} u(0,t) &= 0 \quad t \leq 0 \\ &= U_0(t) \quad t > 0 \end{aligned} \tag{2.5}$$

and,

$$\frac{\partial u(0,t)}{\partial y} = 0 \quad \text{for } t > 0 \tag{2.6}$$

Initially, we presume that:

$$u(y,0)=0 \tag{2.7}$$

Integrating the continuity Equation (2.2), we get v as a constant or function of time. In the present investigation, we take v = 0.

Nondimensional formulation

Let us introduce the nondimensionalization scheme through:

$$\begin{aligned} &u=Uu' \quad y=\frac{\upsilon}{U}y' \quad T-T_\infty=(T_w-T_\infty)\theta \\ &t=\frac{\upsilon}{\alpha U^2}t' \quad \tau_0=\frac{\upsilon}{\alpha U^2}\tau_0' \quad K=\frac{\upsilon^2}{U^2}K' \\ &Q=\frac{Q'U^2(T_w-T_\infty)\lambda}{\upsilon^2} \quad P=\frac{\upsilon c_p\rho}{\lambda} \end{aligned} \tag{2.8}$$

In view of v = 0, with the help of the above nondimensionalization, dropping the primes we get:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}-\left(\frac{eU^2}{\upsilon^3}\right)\frac{\partial^4 u}{\partial y^4}-\left(M_1+\frac{1}{K}\right)u \tag{2.9}$$

$$\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^2\theta}{\partial t^2}=\frac{1}{P}\left[\frac{\partial^2\theta}{\partial y^2}+Q+\tau_0\frac{\partial Q}{\partial t}\right] \tag{2.10}$$

where:

$$M_1=\frac{\sigma B^2\upsilon}{\rho U^2}; \quad M=M_1+\frac{1}{K} \tag{2.11}$$

The boundary conditions to be satisfied by *u* are:

$$\begin{aligned} u(0,t)&=0 \quad t\le 0 \\ &=U_0(t) \quad t>0 \end{aligned} \tag{2.12}$$

$$\frac{\partial u(0,t)}{\partial y}=0 \quad \text{for t} > 0 \tag{2.13}$$

where $U_0(t)$ is a prescribed nondimensional velocity.

Taking the Laplace transform of the equations and the boundary conditions we get:

$$\frac{d^4\bar{u}}{dy^4}=\frac{\upsilon^3}{eU^2}\frac{d^2\bar{u}}{dy^2}-\frac{(s+M)\upsilon^3}{eU^2}\bar{u} \tag{2.14}$$

$$\frac{d^2\bar{\theta}}{dy^2}=ns\bar{\theta}-(1+\tau_0 s)\bar{Q} \tag{2.15}$$

where

$$n(s) = P(1 + \tau_0 s) \tag{2.16}$$

The boundary conditions in the transformed domain are

$$\begin{aligned} &\bar{u}(0, s) = \bar{U}_0(s) \\ &and\ \frac{\partial \bar{u}(0, s)}{\partial y} = 0 \end{aligned} \tag{2.17}$$

and the regularity condition requires that the velocity has to be finite as $y \rightarrow \infty$ which implies that $\bar{u}(y, s)$ is finite as $y \rightarrow \infty$.

3. State space formulation

We choose u, $(\partial u/\partial y)$, $(\partial^2 u/\partial y^2)$, $(\partial^3 u/\partial y^3)$ as the state variables in the physical domain and $\bar{u}(y, s)$, $(d\bar{u}/dy)$, $(d^2\bar{u}/dy^2)$, $(d^3\bar{u}/dy^3)$ as the state variables in the Laplace transform domain.

Let us introduce:

$$\begin{aligned} u_1(y, t) &= \frac{\partial u}{\partial y} \\ u_2(y, t) &= \frac{\partial u_1}{\partial y} \\ u_3(y, t) &= \frac{\partial u_2}{\partial y} \end{aligned} \tag{3.1}$$

Let $\bar{u}$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ denote the Laplace Transforms of u, u_1, u_2, u_3 respectively with respect to t. With this, Equation (2.14) can be rewritten in the form of the following system of first order differential equations:

$$\begin{aligned} \frac{\partial \bar{u}}{\partial y} &= \bar{u}_1(y, s) \\ \frac{\partial \bar{u}_1}{\partial y} &= \bar{u}_2(y, s) \\ \frac{\partial \bar{u}_2}{\partial y} &= \bar{u}_3(y, s) \\ \frac{\partial \bar{u}_3}{\partial y} &= \frac{\upsilon^3}{eU^2}\bar{u}_2(y, s) - \frac{(s + M)\upsilon^3}{eU^2}\ \bar{u}(y, s) \end{aligned} \tag{3.2}$$

This set of equations can be recast in the matrix form as:

$$\frac{d}{dy} = \begin{bmatrix} \bar{u} \\ \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-(s + M)\upsilon^3}{eU^2} & 0 & \dfrac{\upsilon^3}{eU^2} & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} \tag{3.3}$$

This is of the form:

$$\frac{d\bar{V}}{dy} = A(s)\bar{V} \tag{3.4}$$

with,

$$\bar{V}(y,s) = \bar{V}(0,s) \quad for\ y = 0 \tag{3.5}$$

where,

$$\bar{V}(y,s) = (\bar{u}(y,s), \bar{u}_1(y,s), \bar{u}_2(y,s), \bar{u}_3(y,s))^T \tag{3.6}$$

Formal solution of (3.4) and (3.5) is seen to be:

$$\bar{V}(y,s) = \exp[A(s)y]\bar{V}(0,s) \tag{3.8}$$

We obtain the solution $\bar{V}(y,s)$ using the technique of state space analysis.

The characteristic equation of the matrix $A(s)$ is given by:

$$x^4 = \frac{\upsilon^3}{eU^2}x^2 + (s+M)\frac{\upsilon^3}{eU^2} = 0 \tag{3.9}$$

and this can be put in the form where,

$$(x^2 - k_1^2)(x^2 - k_2^2) = 0 \tag{3.10}$$

with,

$$k_1^2 + k_2^2 = \frac{\upsilon^3}{eU^2}; \quad k_1^2 k_2^2 = \frac{(s+M)\upsilon^3}{eU^2} \tag{3.11}$$

$\pm k_1$, $\pm k_2$ are the characteristic roots of $A(s)$ and without loss of generality, we can take k_1, k_2 to be having positive real parts. We note that the matrix $\exp(A(s)y)$ can be put in the Maclaurin's series expansion consisting of I and positive integral power of A (s). The matrix $A(s)$ is a square matrix of order four and it satisfies its characteristic equation which is of fourth degree. Hence A^4 and higher powers of A can be expressed in the form of a matric polynomial of degree three and in view of this, the matrix $\exp(A(s)y)$ takes the form:

$$\exp(A(s)y) = L(y,s) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 \tag{3.12}$$

where $a_0,\ a_1,\ a_2,\ a_3$ are some functions of s and y which are to be determined. From state space theory, we notice that the characteristic roots of $A(s)$ satisfy the equations:

$$\begin{aligned} \exp(k_1 y) &= a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 \\ \exp(-k_1 y) &= a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 \\ \exp(k_2 y) &= a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 \\ \exp(-k_2 y) &= a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 \end{aligned} \tag{3.13}$$

Solving the simultaneous system of Equations (3.13) for $a_0,\ a_1,\ a_2,\ a_3$, we get:

$$
\begin{aligned}
a_0 &= \frac{1}{F}[k_1^2 \cosh(k_2 y) - k_2^2 \cosh(k_1 y)] \\
a_1 &= \frac{1}{F}\left[\left(\frac{k_1^2}{k_2}\right) \sinh(k_2 y) - \left(\frac{k_2^2}{k_1}\right) \quad \sinh(k_1 y)\right] \\
a_2 &= \frac{1}{F}[\cosh(k_1 y) - \cosh(k_2 y)] \\
a_3 &= \frac{1}{F}\left[\left(\frac{1}{k_1}\right) \sinh(k_1 y) - \left(\frac{1}{k_2}\right) \sinh(k_2 y)\right]
\end{aligned}
\tag{3.14}
$$

Substituting these in (3.5) we can evaluate the elements L_{ij} of the matrix:

$$L(y,s) = \exp(A(s)y)$$

These are given by:

$$
\begin{aligned}
L_{11} &= \frac{1}{F}[k_1^2 \cosh(k_2 y) - k_2^2 \cosh(k_1 y)] \\
L_{12} &= \frac{1}{F}\left[\frac{k_1^2}{k_2}\sinh(k_2 y) - \frac{k_2^2}{k_1}\sinh(k_1 y)\right] \\
L_{13} &= \frac{1}{F}[\cosh(k_1 y) - \cosh(k_2 y)] \\
L_{14} &= \frac{1}{F}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right] \\
L_{21} &= \frac{-k_1^2 k_2^2}{F}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right] \\
L_{22} &= \frac{1}{F}[k_1^2 \cosh(k_2 y) - k_2^2 \cosh(k_1 y)] \\
L_{23} &= \frac{1}{F}[k_2 \sinh(k_2 y) - k_1 \sinh(k_1 y)] \\
L_{24} &= \frac{1}{F}[\cosh(k_1 y) - \cosh(k_2 y)] \\
L_{31} &= \frac{-k_1^2 k_2^2}{F}[\cosh(k_1 y) - \cosh(k_2 y)] \\
L_{32} &= \frac{-k_1^2 k_2^2}{F}\left[\frac{1}{k_1}\sinh(k_1 y) - \frac{1}{k_2}\sinh(k_2 y)\right] \\
L_{33} &= \frac{1}{F}[k_1^2 \cosh(k_1 y) - k_2^2 \cosh(k_2 y)] \\
L_{41} &= \frac{k_1^2 k_2^2}{F}[k_1 \sinh(k_1 y) + k_2 \sinh(k_2 y)] \\
L_{42} &= \frac{-k_1^2 k_2^2}{F}[\cosh(k_1 y) - \cosh(k_2 y)]
\end{aligned}
$$

$$L_{43} = \frac{1}{F}[k_1^3 \sinh(k_1 y) - -k_2^3 \sinh(k_2 y)]$$
$$L_{44} = \frac{1}{F}[k_1^2 \cosh(k_1 y) - k_2^2 \cosh(k_2 y)] \tag{3.15}$$

where,

$$F = k_1^2 - k_2^2$$

The physics of the problem indicates that far away from the plate (i.e.) as $y \rightarrow \infty$, the disturbance caused by the impulsive motion of the plate $y = 0$ is not felt and due to this, the solution has to be finite. Hence we must replace sinh (ky) and cosh (ky) terms in L_{ij} 's by $-e^{-ky}/2$ and $e^{-ky}/2$, respectively. With this change, let the matrix L be denoted by L^*. Hence we can obtain the solution in the Laplace transform domain as:

$$\bar{V}(y, s) = L^*(y, s)\bar{V}(0, s) \tag{3.16}$$

In the vector,

$$\bar{V}(0, s) = (\bar{u}(0, s), \bar{u}_1(0, s), \bar{u}_2(0, s), \bar{u}_3(0, s))$$

we know that,

$$\bar{u}(0, s) = \bar{U}_0(s)$$
$$\bar{u}_1(0, s) = 0$$

and $\bar{u}_2(0, s)$, $\bar{u}_3(0, s)$ are unknowns. We obtain these by substituting $y = 0$ in (3.16) to get a system of two equations in the unknowns $\bar{u}_2(0, s)$, and $\bar{u}_3(0, s)$. From this system, we get:

$$\bar{u}_2(0, s) = -k_1 k_2 \bar{U}_0$$
$$\bar{u}_3(0, s) = k_1 k_2 (k_1 + k_2)\bar{U}_0 \tag{3.17}$$

Thus we get the Laplace transform of the velocity component $\bar{u}(y, s)$ as:

$$\bar{u}(y, s) = \frac{k_1 e^{-k_2 y} - k_2 e^{-k_1 y}}{(k_1 - k_2)} \bar{U}_0(s) \tag{3.18}$$

4. Solution of the energy equation

The equation of energy given in (2.10), after taking the Laplace transform, is transformed to:

$$\frac{d^2\bar{\theta}}{dy^2} = ns\bar{\theta} - (1 + \tau_0 s)\bar{Q} \tag{4.1}$$

where,

$$n(s) = P(1 + \tau_0 s) \tag{4.2}$$

and $\bar{\theta}$ and $\bar{Q}$ are Laplace transforms of θ and Q, respectively. We shall consider two applications for illustration:

Case (i)

Let us consider the flow of an incompressible conducting couple stress fluid in a porous medium occupying a semi infinite region $y \geq 0$ of space bounded by a moving plate $y = 0$. Let us assume that a thermal shock of the form:

$$\theta(0, t) = \theta_0 H(t) \tag{4.3}$$

is applied to the plane at time $t = 0$ where θ_0 is a constant and $H(t)$ is Heaviside unit step function. Assuming all initial conditions to be 0 and taking Laplace transform of (4.3), we see that $\bar{\theta}$ is given by:

$$\frac{d^2\bar{\theta}}{dy^2} = ns\bar{\theta} \text{ and } \bar{\theta}(0, s) = \frac{\theta_0}{s} \tag{4.4}$$

Hence $\bar{\theta}(y, s)$

$$\bar{\theta}(y, s) = \frac{\theta_0 e^{-\sqrt{ns}y}}{s} \tag{4.5}$$

where $n = n(s) = P(1 + \tau_0 s)$ and in view of this exact numerical inversion seems to be difficult.

Case (ii)

Here, instead of the thermal shock taken in case (i), we assume that there is a plane distribution of continuous heat sources located at the plate at y $=$ 0. We shall take the intensity of the heat sources as:

$$Q(y, t) = Q_0 H(t)\delta(y) \tag{4.6}$$

where Q_0 is a constant and δ (y) is Dirac delta function. Taking Laplace transform of (4.6) we get:

$$\bar{Q}(y, s) = \frac{Q_0 \delta(y)}{s} \tag{4.7}$$

Let us consider a right circular cylinder of unit base with its axis perpendicular to the plane source of heat and whose bases lie on opposite sides of it. We use Gauss divergence theorem to obtain the thermal condition at the plane source. Taking the limit as the height of the cylinder tends to 0 and noting that there is no heat flux through the lateral surface of the cylinder, we get:

$$q(0, t) = \frac{Q_0}{2} H(t) \tag{4.8}$$

Using the generalized Fourier law of heat conduction in the nondimensional form, namely:

$$q + \tau_0 \frac{\partial q}{\partial t} = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0} \qquad (4.9)$$

and taking the Laplace transform for this, after some algebra, we get:

$$\bar{\theta}'(0, s) = -\frac{nQ_0}{2Ps} \qquad (4.10)$$

In this case we finally get:

$$\bar{\theta}(y, s) = \frac{\sqrt{n}Q_0}{2Ps^{\frac{3}{2}}} e^{-\sqrt{ns}y} + \frac{Q_0}{Ps^2} \qquad (4.11)$$

Introducing (θ/Q_0) as nondimensional temperature distribution, $\bar{\theta}$ in nondimensional form is given by:

$$\bar{\theta}(y, s) = \frac{\sqrt{n}}{2Ps^{\frac{3}{2}}} e^{-\sqrt{ns}y} + \frac{1}{Ps^2} \qquad (4.12)$$

5. Inversion of $\bar{u}(y, s)$ and $\bar{\theta}(y, s)$

In section 3 we have obtained the Laplace transform of $u(y,t)$ through Equation (3.18) and in section 4, we obtained the Laplace transform $\bar{\theta}(y, s)$ of the temperature distribution in the case of thermal shock problem through (4.5) and in the case of plane continuous distribution of heat sources located at $y = 0$ through Equation (4.12). The Laplace transforms involve s and the parameters k_1, k_2 and n which are functions of s. Direct inversion of these Laplace transforms is difficult in view of the involved nature of the functions $\bar{u}(y, s)$ and $\bar{\theta}(y, s)$. Hence we propose to invert these functions by the numerical inversion procedure introduced by Honig and Hirdes (1984).

To invert the Laplace transform in the above equations, a numerical technique based on the Fourier expansion of a function is used.

Let $\bar{f}(s)$ be the Laplace transform of a given function $f(t)$. The inversion formula for the Laplace transform states that:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{iyt} \bar{f}(s) ds \qquad (5.1)$$

where c is an arbitrary constant greater than all the real parts of the singularities of $\bar{f}(s)$. Taking $s = \mathrm{c} + \mathrm{i}y$, we get:

$$f(t) = \frac{e^{ct}}{2\pi} \int_{c-i\infty}^{c+i\infty} e^{iyt} \bar{f}(c + iy) dy$$

This integral can be approximated by:

$$f(t) = \frac{e^{ct}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y} \bar{f}(c + ik\Delta y)\ \Delta y$$

Taking $\Delta y = \pi/t_1$, we get:

$$f(t) = \frac{e^{ct}}{t_1} \mathrm{Re}\left[\frac{1}{2}\bar{f}(c) + \sum_{k=1}^{\infty} e^{ik\Pi t/t_1} \bar{f}\left(c + \frac{ik\pi}{t_1}\right)\right]$$

For numerical purposes, this is approximated by the function:

$$f_N(t) = \frac{e^{ct}}{t_1} \mathrm{Re}\left[\frac{1}{2}\bar{f}(c) + \sum_{k=1}^{N} e^{ik\Pi t/t_1}\ \bar{f}\left(c + \frac{ik\pi}{t_1}\right)\right] \tag{5.2}$$

where N is a sufficiently large integer chosen such that:

$$f(t) = e^{ct} \mathrm{Re}\left[e^{iN\Pi t/t_1} \bar{f}\left(c + \frac{iN}{t_1}\right)\right] < \varepsilon \tag{5.3}$$

and ε is a preselected small positive number that corresponds to the degree of accuracy to be achieved. Formula (5.2) is the numerical inversion formula valid for $2t_1 \geq t \geq 0$.

In particular, we choose $t = t_1$, obtaining:

$$f_N(t) = \frac{e^{ct}}{t} \mathrm{Re}\left[\frac{1}{2}\bar{f}(c) + \sum_{k=1}^{N} (-1)^k\ \bar{f}\left(c + \frac{ik\pi}{t_1}\right)\right] \tag{5.4}$$

6. Numerical discussion

$u(y,t)$ and $\theta(y,t)$ denote the inverse Laplace Transforms of $\bar{u}(y,s)$ *and* $\bar{\theta}(y,s)$, respectively. To determine these we attempt the problem with $U_0(t) = 1$ for all $t > 0$. With this $\bar{U}_0(s) = 1/s$. $u(y,t)$ and $\theta(y,t)$ are obtained for various values of y and t for diverse values of c_1, M, R and different values of y and t through the cited numerical inversion procedure. In Figure 1 we plot the variation of velocity with respect to distance y for different values of time for fixed values of the other parameters indicated in the figure. As expected, as y increases, there is a decrease in u.

Figure 2 displays the variation of velocity with respect to y for different values of c_1 for fixed time $t = 1$, $M = 2$ and $R = 0.5$. As the couple stress parameter increases the velocity shows a decreasing trend for any fixed y. Thus a decrease in couple stress viscosity leads to an increase in the fluid velocity. For any fixed c_1, as $y \to \infty$, the velocity tends to 0.

Figure 3 shows the variation of velocity with respect to the parameter M involving the magnetic effect and thermal conductivity. As M increases the velocity decreases for any fixed y. Further velocity tends to 0 as $y \to \infty$.

Figure 4 describes the variation of velocity with respect to the Reynolds number R. An increase in R indicates a decrease in μ. As R increases, the velocity increases. Further for a fixed R, velocity tends to 0 as $y \to \infty$ as in the other cases.

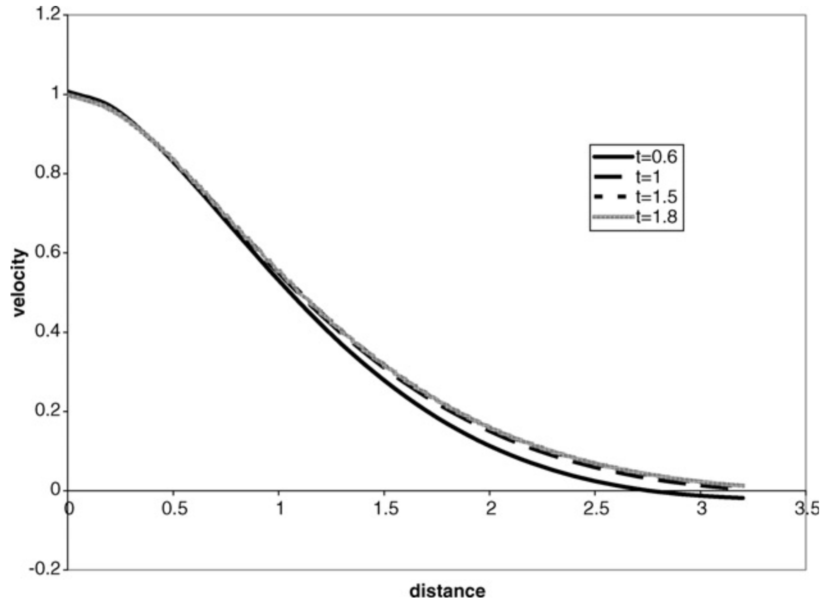

Figure 1.
Variation of velocity for $c_1 = 0.5$, $M = 2$, $R = 0.5$ as the parameter t changes

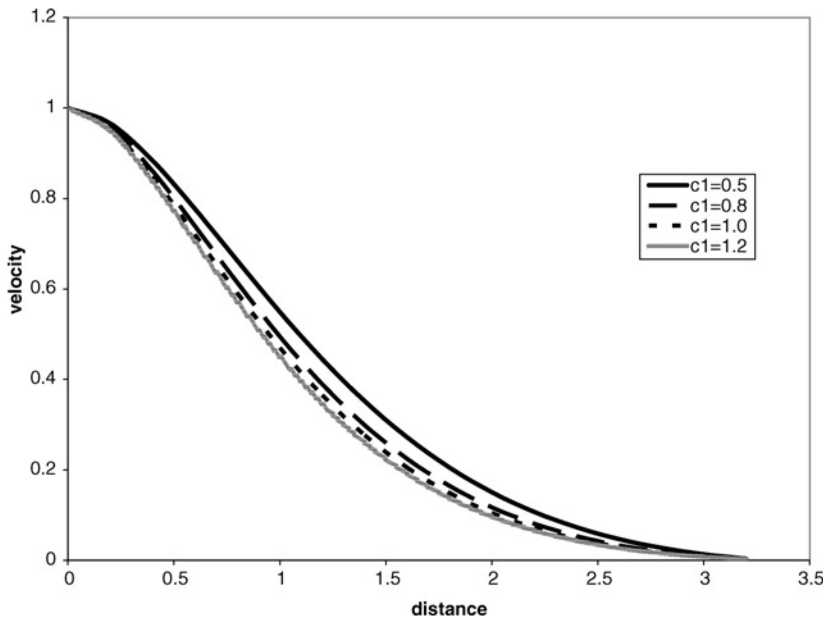

Figure 2.
Variation of velocity for $t = 1$, $M = 2$, $R = 0.5$ as the parameter c_1 changes

In Figure 5 we plot the temperature distribution $\theta(y, t)$ in the thermal shock problem considered in case (i) of section 5. As the relaxation time τ_0 (written as $t1$ in Figure 5) takes values 0.0, 0.1, 0.2 the temperature tends to 0 as $y \to \infty$.

In Figure 6, in the case of plane continuous temperature distribution on the plate $y = 0$. (Case (ii) of section 5), the temperature is seen to increase initially for any

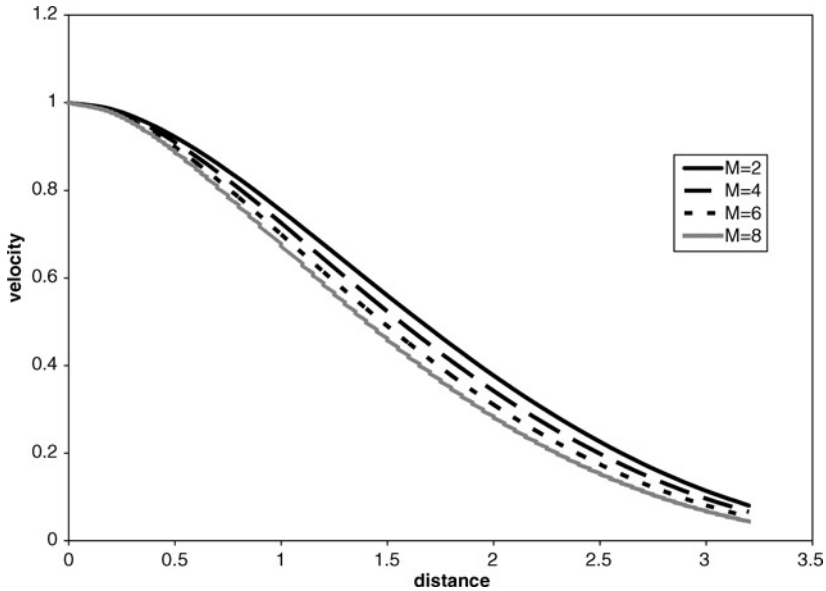

Figure 3.
Variation of velocity for $t = 1$, $c_1 = 0.5$, $R = 0.5$ as the parameter M changes

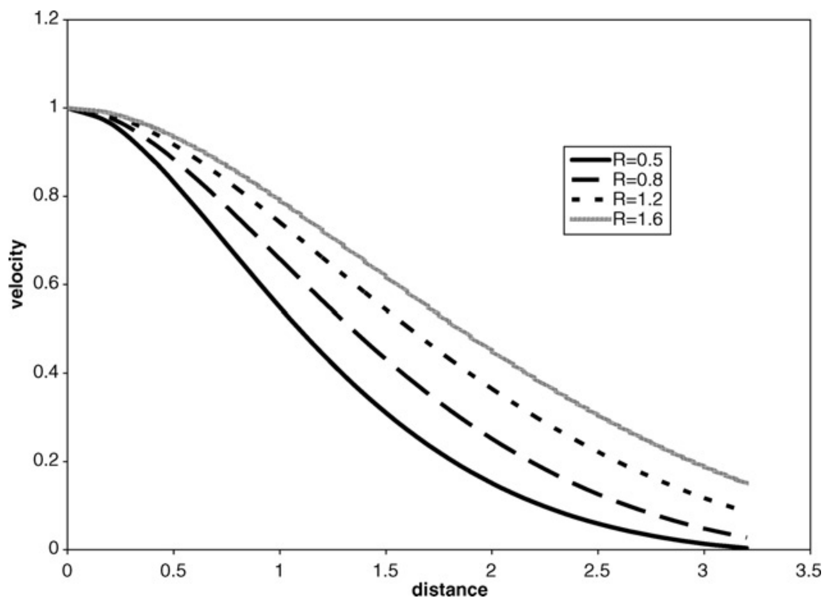

Figure 4.
Variation of velocity for $t = 1$, $c_1 = 0.5$, $M = 2$, as the parameter R changes

$\tau_0 = t1$ and then found to be tending to 0 as $y \to \infty$. Further, nearer to the plate, as τ_0 increases, θ increases as y increases from 0 to a critical value and then quickly tends to 0 as $y \to \infty$.

7. Conclusion

The problems in couple stress fluid flows earlier have been solved by conventional methods either analytically or numerically. In the present paper the couple stress fluid

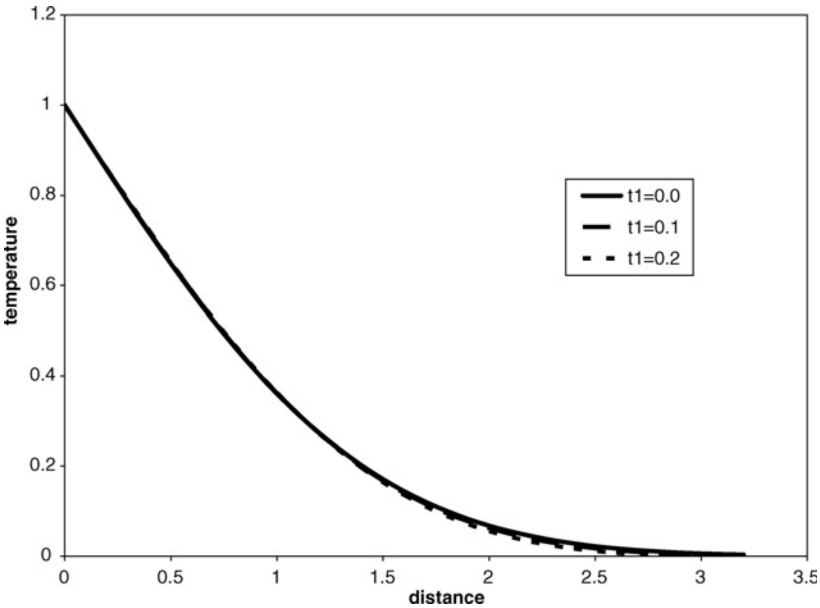

Figure 5.
Variation of temperature for different values of $t1$ (thermal shock problem)

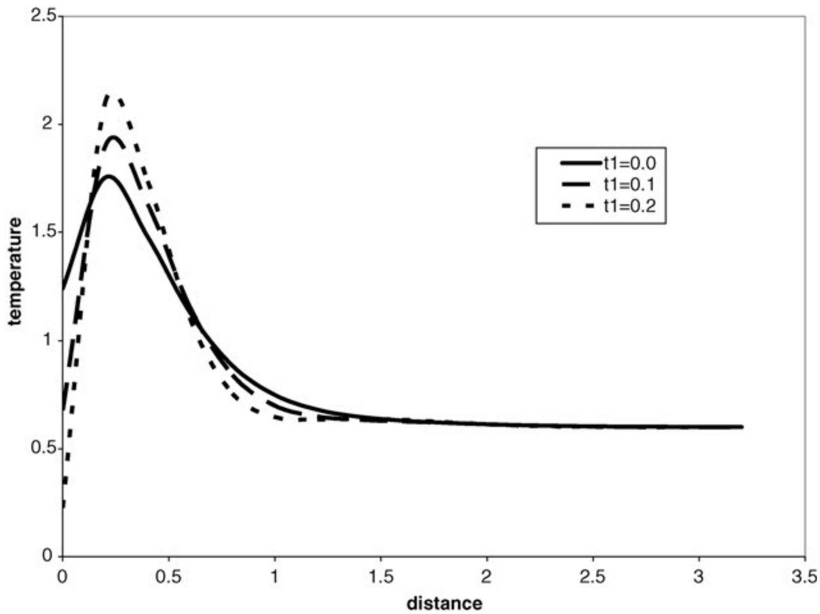

Figure 6.
Variation of temperature for different values of $t1$ (continuous temperature distribution)

flow equations introducing magnetic effect pertaining to free convection flow of the fluid through a porous medium over an infinite plate that is started into motion in its own plane by an impulse is considered. Deviating from the classical approach we have adopted a state space approach to solve the problem. This approach which is used frequently in modern control theory is adopted here and this method gives exact solutions for the velocity and temperature distributions in Laplace transform domain. We notice that as the intensity σ of the magnetic field increases, it results in the increase of the parameter M. As M increases, the velocity decreases. In fact, an increase in σ leads to an increase in M, which results in an increase of the Lorentez force $\bar{F} = \sigma(\bar{u} \times \bar{B}) \times \bar{B}$. This force increases the retardation and opposes the flow and hence the decrease in the velocity. This is in agreement with our expectation that an imposition of magnetic field results in a decrease of velocity.

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Corresponding author
G. Shantha can be contacted at: Shantha_maths@yahoo.com